APPROXIMATE METHOD FOR DETERMINING NONSTATIONARY ONE-DIMENSIONAL TEMPERATURE FIELDS

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An approximate method for solution of the heat-conduction equation is considered; it can be used to reduce a boundary-value problem for a partial-differential equation to a Cauchy problem for a system of ordinary differential equations. A generalization to a problem with unknown boundary is given.

We shall find the temperature field T(x, t) in a plate at whose surface x = 0 and x = l the heat fluxes $q_1(t)$ and $q_2(t)$ are specified; $q_1(0) = q_2(0) = 0$, while $T(x, 0) = T_0$. If the thermal characteristics of the plate material are constant, the solution can be found by separation of variables [1]:

$$\frac{\theta(\xi,\tau)}{B} = -q_1(\tau)\left(\xi - \frac{\xi^2}{2}\right) + q_2(\tau)\frac{\xi^2}{2} - \frac{q_2(\tau)}{6} + \frac{q_1(\tau)}{3}$$
$$+ f^2\int_0^\tau [q_1(\eta) + q_2(\eta)]\,d\eta - 2\sum_{n=1}^{\infty}\frac{1}{(n\pi)^2}\int_0^\tau [\dot{q}_1(\eta) + (-1)^n\dot{q}_2(\eta)]$$
$$\times \exp[-f^2(n\pi)^2(\tau-\eta)]\,d\eta\cos n\pi\xi, \tag{1}$$

where

$$\xi = \frac{x}{l};\quad \theta = \frac{T-T_0}{T_0};\quad \tau = \frac{t}{t_k};\quad f^2 = \frac{at_k}{l^2};\quad a = \frac{\lambda}{\rho c};$$
$$B = \frac{l}{\lambda T_0};\quad \dot{q} = \frac{dq}{dt}.$$

We assume that the exact temperature value $\theta(\xi, \tau)$ at the beginning of the time interval $0 \le \tau \le \tau_0$ is of no interest (the temperature may still be small, for example); when $\tau_0 \le \tau \le 1$, however, it is necessary to obtain a fairly exact solution. Let the heat fluxes be specified as

$$q_1(\tau) = \sum_{i=1}^{n} a_i\tau^i,\quad q_2(\tau) = 0. \tag{2}$$

We introduce the "approximate solution"

$$\frac{\theta_1(\xi,\tau)}{B} = -q_1(\tau)\left(\xi - \frac{\xi^2}{2}\right) + q_2(\tau)\frac{\xi^2}{2} - \frac{q_2(\tau)}{6} + \frac{q_1(\tau)}{3} + f^2\int_0^\tau (q_1+q_2)\,d\eta \tag{3}$$

and the "refined solution"

$$\frac{\theta_2(\xi,\tau)}{B} = \frac{\theta_1(\xi,\tau)}{B} + \frac{2}{\pi^2}\int_0^\tau [\dot{q}_2(\eta) - \dot{q}_1(\eta)]\ \exp[-f^2\pi^2(\tau-\eta)]\,d\eta\cos\pi\xi. \tag{4}$$

On the assumption that $q_1(\tau) = a_n\tau^n$, we compare the results obtained from (1), (3), and (4). Figure 1 shows graphs of the functions $(\theta_1-\theta)/\theta$ and $(\theta_2-\theta)/\theta$ for various n, f^2, and $\xi = 0$. When n = 1 and $f^2 > 1$, when $\tau > 0.2$, the accuracy of (4) is fully satisfactory for practical purposes. When $f^2 > 6$–7 and $\tau > 0.2$,

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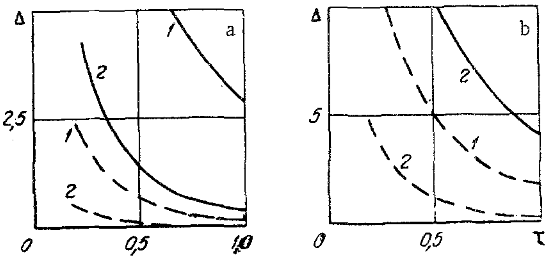

Fig. 1. Estimate for accuracy of approximate relationships: a) n = 1; b) n = 5; τ) dimensionless time; Δ) error in percent; solid lines) approximate solution; dashed lines) refined solution; curves 1, 2 correspond to $f^2 = 1$; $f^2 = 3$.

Eq. (3) is also accurate enough. As n increases, the accuracy of (3) and (4) drops. In this case, however, the term $a_n\tau^n$ in (2) makes a significant contribution only when τ approaches 1. Within this region of variation of τ, however, Eq. (4) is acceptably accurate. Thus for an external action of very general type, the approximate relationships (3)-(4) are sufficiently accurate over a wide range of definition of the solution.

Let us use these relationships to solve various boundary-value problems. In (4) we go over to physical variables, calculating the surface temperatures $T(0, t) \equiv T_1(t)$ and $T(l, t) \equiv T_2(t)$, and eliminating the integral terms in the resulting expressions by differentiating with respect to t. We obtain

$$\dot{T}_1 + \dot{T}_2 = \frac{l}{6\lambda}(\dot{q}_1 + \dot{q}_2) + \frac{2}{\rho c l}(q_1 + q_2);$$
$$\dot{T}_1 - \dot{T}_2 = \frac{lk}{6\lambda}(\dot{q}_1 - \dot{q}_2) - \frac{\alpha a \pi^2}{l^2}(T_1 - T_2) + \frac{\pi^2 \alpha}{2\rho c l}(q_1 - q_2), \tag{5}$$
$$k = 3 - \frac{24}{\pi^2}.$$

When $\alpha = 0$ and $\alpha = 1$, we obtain results corresponding to (3) and (4).

To the equations obtained we add the boundary conditions

$$q_1 = f_1(T_1, t); \quad q_2 = f_2(T_2, t). \tag{6}$$

Equations (5) and (6) form a system of four equations in four unknown functions $q_1(t)$, $q_2(t)$, $T_1(t)$, $T_2(t)$. The initial conditions are $T_1(0) = T_2(0) = 0$. For boundary conditions of the first kind, the known functions $T_1(t)$ and $T_2(t)$ are solved, and we solve two equations of (5) for $q_i(t)$ under the zero initial conditions. In both cases, therefore, the problem reduces to solution of a Cauchy problem for a system of ordinary differential equations.

To determine the temperature field of a two-layer (multilayer, in general) plate, Eqs. (5) must be written for each of the plates, with allowance for the continuity of the temperature field at the common boundary. Then to determine the temperatures T_1 and T_2 of the surfaces, the temperature T_3 of the common boundary, and the heat flux q_3 through it, we have a system of four differential equations that must be supplemented by boundary conditions.

As we see from Fig. 1, the accuracy of (5) rises rapidly with f^2. Thus we can use the following approach to refine the solution. We mentally divide the homogeneous plate into two parts with thicknesses $l/2$, and employ the equations for determining the temperature field in a two-layer plate. For each part, the criterion f^2 rises by a factor of 4, and the accuracy of Eqs. (5), applied to each of the subplates, increases sharply.

This method can be used to determine the temperature field of a plate at whose surface phase transformations occur. As we see from Fig. 1, the accuracy of Eqs. (5) rises rapidly with the time, and after a certain time Δt following the beginning of heating, we can assume that (5) is exact. Let us assume that the physical conditions are such that over the characteristic time Δt the plate thickness varies by an amount Δl owing to sublimation of the surface layer

$$\Delta l / l \ll 1. \tag{7}$$

In this case, in (5) we can replace the constant thickness l by the average value $\bar{l}$ over the time Δt, with different time intervals corresponding to different $\bar{l}$. Thus in (5), the thickness l can be treated as a new unknown time function. In place of the single condition (6) at the sublimating surface, we have the two conditions

$$T_1(t) = T_s = \text{const}; \quad q_1(t) = f_1(T_1, t) + \rho \dot{l}(\mu + \chi). \tag{8}$$

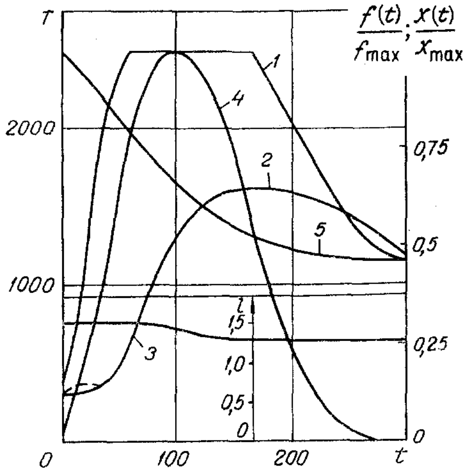

Fig. 2. Results of solution of Eqs. (5), (9), (10); T) temperature, °K; t) time, sec; l) thickness of shield, cm; curves: 1) $T_1(t)$; 2) $T_2(t)$; 3) $l(t)$; 4) $f(t)/f_{max}$; 5) $\chi(t)/\chi_{max}$; dashed lines) $T_2(t)$ for solution without division of plate into two parts.

Equations (5), (8), and the second equation of (6) form a system of five equations for determining the five functions T_1, T_2, q_1, q_2, l of the time t. The transition from the solution of (5)-(6) to the system (5), (8), (6) occurs at the time t_1 at which $T_1(t_1) = T_s$. The inverse transition, if it occurs, takes place at time t_2, when $l(t_2) = 0$. After substitution of (8) into (5) for $l(t)$, we obtain a second-order equation. The missing initial condition $\dot{l}(t_1) = \dot{l}_0$ is ordinarily easy to provide by analyzing the physical nature of the problem.

As an example, let us look at the heating of a protective heat shield made of a destructible heat-insulating material, when a spacecraft enters the denser layers of the atmosphere [2]. At the outer surface of the shield we have

$$q_1(t) = f(t) - \sigma\varepsilon_1 T_1^4, \quad T_1 < T_s;$$
$$q_1(t) = f(t) - \sigma\varepsilon_1 T_p^4 + \rho\dot{l}[\mu + \chi(t)], \quad T_1 = T_s. \tag{9}$$

If heat is transferred radiatively from the inside surface of the shield to the surface of the inner layer of heat insulation [2], where the temperature varies little, then

$$q_2 = -\sigma\varepsilon_2(T_2^4 - T_0^4). \tag{10}$$

The functions f(t) and $\chi(t)$ are determined by the trajectory descent. The system (5), (9)-(10) was solved on a high speed electronic computer by the Runge–Kutta method. Figure 2 shows a graph of the functions f(t), $\chi(t)$, $l(t)$, $T_1(t)$, $T_2(t)$ for the following data:

$$f(t) = A(1-\tau)^4\tau^2; \quad \max f(t) = 3.36 \ \text{MV/m}^2; \ \varepsilon_1 = 0.8; \ \varepsilon_2 = 0.4;$$
$$\chi(t) = 23.4 + 26.8(1-\tau)^4(\tau+1)^2, \quad \text{MJ/kg}; \ t_t = 300 \ \text{sec};$$
$$l = 0.015 \ \text{m}; \ \tau = t/t_t.$$

The properties of the hypothetical shield materials are specified by the following parameters: $\rho = 1350$ kg/m^3; $\lambda = 2.1$ W/m·deg; c = 1.47 kJ/kg·deg; $\mu = 1$ MJ/kg; $T_S = 2500$°K; $T_0 = 300$°K.

In form, the functions f(t), $\chi(t)$ correspond roughly to the first penetration of a spacecraft into the earth's atmosphere at the second cosmic speed [3].

Comparing the solutions (I, without partitioning of the plate into two parts; II, the refined solution with fictitious partitioning of the plate) we see that except for the initial 25 sec long segment, solutions I and II are nearly the same. When $l > 2.0$ cm, it is necessary to partition the plate in order to refine the solution. When $l > 4.0$ cm, even this technique does not provide adequate accuracy, and other solution methods must be employed.

The machine time required to solve one version of the problem with $\delta t = 2$ sec does not exceed 30 sec, which reduces machine time requirements by a factor of 30-40 as compared with the finite-difference dispersion method.

NOTATION

x, ξ	are dimensioned and dimensionless coordinates;
t, τ	are dimensioned and dimensionless time;
t_t	is the duration of the thermal effect;
l	is the thickness of the plate;
λ, ρ, c, a	are the thermal conductivity, density, specific heat capacity, and thermal diffusivity of the material;
$T(x, t)$, T_0	are the instantaneous and initial temperatures of the plate;
θ	is the dimensionless temperature;
$\theta(\xi, \tau)$, $\theta_1(\xi, \tau)$, $\theta_2(\xi, \tau)$	are the exact, approximate, and refined solutions in dimensionless form;
$q_1(t)$, $q_2(t)$, $q_3(t)$	are the heat fluxes acting on the plate surface;

$T_1(t)$, $T_2(t)$, $T_3(t)$ are the temperatures of the plate surfaces;
a_i is the coefficient for the expansion of $q_1(t)$ into series in powers of t;
n is the exponent in the polynomial used to approximate the heat flux q_1;
T_S is the sublimation temperature of the material;
μ is the heat of sublimation;
$f_1(T_1, t), f(t), \chi(t)$ are known functions;
σ is the Boltzmann constant;
ε_1, ε_2 are the emissivities of the radiating surfaces;
δt is the integration state;
A is a coefficient that determines the value of the heat flux.

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